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Equivariant characteristic forms on the bundle of connections

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Abstract

The characteristic forms on the bundle of connections of a principal bundle $P \to M$ of degree equal to or less than $\dim M$, determine the characteristic classes of P, and those of degree $k + \dim M$ determine certain differential k-forms on the space of connections $\mathcal{A}$ on P.

The equivariant characteristic forms provide canonical equivariant extensions of these forms, and therefore canonical cohomology classes on $\mathcal{A}/\mathrm{Gau}^0 P$. More generally, for any closed $\beta \in \Omega^r(M)$ and $f \in \mathcal{I}_k^G$, with $2k + r \geq \dim M$, a cohomology class on $\mathcal{A}/\mathrm{Gau}^0 P$ is obtained. These classes are shown to coincide with some classes previously defined by Atiyah and Singer.
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1. Introduction

Let $\pi : P \to M$, be a principal G-bundle and let $p : C(P) \to M$ be its bundle of connections. Let $\mathcal{I}_k^G$ be the space of Weil polynomials of degree k for G. The principal G-bundle

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$C(P) \times_M P \to C(P)$ is endowed with a canonical connection $\mathbb{A}$ (see below for the details), which can be used to obtain, for every $f \in \mathcal{I}_k^G$, a characteristic $2k$-form on $C(P)$, denoted by $c_f(\mathbb{F}) = f(\mathbb{F}, \ldots^{(k}, \mathbb{F})$ (e.g., see [11]), where $\mathbb{F}$ is the curvature of $\mathbb{A}$. Moreover, such a form is closed and Aut P-invariant. As $C(P)$ is an affine bundle, the map $p^* : H^*(M) \to H^*(C(P))$ is an isomorphism. The cohomology class in M corresponding to $c_f(\mathbb{F})$ under this isomorphism is the characteristic class of P associated to f. Hence, the characteristic forms on $C(P)$ determine the characteristic classes on M, but the characteristic forms contain more information than the characteristic classes; for example, the characteristic classes of degree $2k > n$ vanish, although the corresponding forms do not necessarily, as $\dim C(P) > \dim M$. Precisely, the principal aim of this paper is to provide a geometric interpretation of such characteristic forms of higher degree.

This is based on the following construction. Let $E \to N$ be an arbitrary bundle over a compact, oriented n-manifold without boundary. We define a map $\mathcal{F} : \Omega^{n+k}(J^r E) \to \Omega^k(\Gamma(E))$ commuting with the exterior differential and with the action of the group $\mathrm{Proj}^+(E)$ of projectable diffeomorphisms which preserve the orientation on M. Hence, if $\alpha \in \Omega^{n+k}(J^r E)$ is closed, exact, or invariant under a subgroup $\mathcal{G} \subset \mathrm{Proj}^+(E)$, then the form $\mathcal{F}[\alpha]$ enjoys the same property.

Applying this construction to the bundle $C(P) \to M$, for any characteristic form $c_f(\mathbb{F})$ with $2k > n$, we obtain a closed and Gau P-invariant $(2k - n)$-form on the space $\mathcal{A} = \Gamma(M, C(P))$ of connections on P. More generally, as proved in [11], the space of Gau P-invariant forms on $C(P)$ is generated by forms of type $c_f(\mathbb{F}) \wedge p^*\beta$, with $\beta \in \Omega^*(M)$. So, given $f \in \mathcal{I}_k^G$ and a closed $\beta \in \Omega^r(M)$, such that $2k + r \geq n$, we have a closed and Gau P-invariant $(2k + r - n)$-form on $\mathcal{A}$ given by

$$C_{f,\beta} = \mathcal{F}[c_f(\mathbb{F}) \wedge p^*\beta] \in \Omega^{2k+r-n}(\mathcal{A}). \tag{1}$$

As $\mathcal{A}$ is an affine space, these forms are exact, and the cohomology classes defined by them on $\mathcal{A}$, vanish; but in gauge theories—because of gauge symmetry—it is more interesting to consider the quotient space $\mathcal{A}/\mathrm{Gau}\, P$ instead of the space $\mathcal{A}$ itself. Although the forms (1) are Gau P-invariant, they are not projectable with respect to the natural quotient map $\mathcal{A} \to \mathcal{A}/\mathrm{Gau}\, P$. Hence they do not define directly cohomology classes on $\mathcal{A}/\mathrm{Gau}\, P$. Consequently, we are led to consider another way in order to obtain cohomology classes on the quotient from these forms. As is well known, the cohomology of the quotient manifold by the action of a Lie group, is related to the equivariant cohomology of the manifold, e.g., see [19]. Below, we show that the usual construction of equivariant characteristic classes (e.g., see [6,7,9]) when applied to the canonical connection $\mathbb{A}$, provides canonical Aut P-equivariant extensions of the characteristic forms. By extending the map $\mathcal{F}$ to equivariant differential forms in an obvious way, this result allows us to obtain Gau P-equivariant extensions of the forms (1); see Theorem 16 below. These extensions determine cohomology classes in the quotient space $\mathcal{A}/\mathrm{Gau}^0 P$, where $\mathrm{Gau}^0 P \subset \mathrm{Gau}\, P$ is the subgroup of gauge transformations preserving a fixed point $u_0 \in P$. We also prove that such classes coincide with those defined in [3].

As is well known (e.g. see [2]), an equivariant extension of an invariant symplectic two-form is equivalent to a moment map for it. Hence, if the form (1) is of degree two on $\mathcal{A}$, then the Gau P-equivariant extension that we obtain, defines a canonical moment map for the symplectic action of the gauge group on $\mathcal{A}$, and we show that these symplectic forms and moment maps coincide with those defined in [1,13,21].

Finally we show how our constructions lead to conservation laws for the Chern–Simons terms considered in [18].

2. The bundle of connections and the canonical connection

If $\pi : P \to M$ is a principal G-bundle, its bundle of connections is an affine bundle $p : C(P) \to M$ modelled over the vector bundle $T^*M \otimes \operatorname{ad} P$, such that there is a bijection between connections on P and the sections of $C(P)$ (e.g. see [10,16,22]). The natural projection $\bar{p} : \mathbb{P} = C(P) \times_M P \to C(P)$ onto the first factor induces a principal G-bundle structure over $C(P)$, and we have the commutative diagram

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{\bar{p}} & P \\ \bar{\pi} \downarrow & & \downarrow \pi \\ C(P) & \xrightarrow{p} & M \end{array}$$

The bundle $\mathbb{P}$ has a canonical connection $\mathbb{A} \in \Omega^1(\mathbb{P}, \mathfrak{g})$ characterized by,

$$\mathbb{A}_{(\sigma_A(x),u)}(X) = A_u(\bar{p}_* X) \tag{2}$$

for every connection A on P, $x \in M$, $u \in \pi^{-1}(x)$, $X \in T_{(\sigma_A(x),u)}\mathbb{P}$, and where $\sigma_A : M \to C(P)$ is the section corresponding to A.

Remark 1. It can be shown (see [10]) that the bundle $\bar{p} : \mathbb{P} \to C(P)$ is isomorphic to $J^1P \to (J^1P)/G$ and, under this identification, the canonical connection $\mathbb{A}$ corresponds to the structure form of J^1P.

The canonical connection enjoys the following properties (e.g. see [10]):

(1) $\mathbb{A}$ is invariant under the natural action of the group $\operatorname{Aut} P$ of automorphisms of P.
(2) For every connection A on P, we have $\bar{\sigma}_A^*(\mathbb{A}) = A$, where $\sigma_A : P \to \mathbb{P}$ is defined by $\bar{\sigma}_A(u) = (\sigma_A(x), u)$, with $x \in M$, $u \in \pi^{-1}(x)$.

Let $\mathbb{F}$ be the curvature of $\mathbb{A}$. If $f \in \mathcal{I}_k^G$ is a Weil polynomial of degree k for G, we define the characteristic form associated to f as the $2k$-form on $C(P)$ defined by $c_f(\mathbb{F}) = f(\mathbb{F}, \ldots, \mathbb{F})$. This form has the following properties:

(3) $c_f(\mathbb{F})$ is closed.
(4) $c_f(\mathbb{F})$ is invariant under the action of the group $\operatorname{Aut} P$ on $C(P)$.
(5) For every connection A on P we have $\sigma_A^*(c_f(\mathbb{F})) = f(F_A, \ldots, F_A)$.

As a consequence of (3) and (5) and the fact that the space of connections is an affine space, we obtain the well-known result of Chern-Weil theory that the cohomology class $[f(F_A,\dots,F_A)]\in H^{2k}(M)$ is independent of the connection A, and is it called the characteristic class associated to f. In other words, the map σ_A^* is an inverse of $p^*: H^\bullet(M)\to H^\bullet(C(P))$, and under this isomorphism the cohomology class of $c_f(\mathbb{F})$ corresponds to the characteristic class of P associated to f (e.g. see [11,22]).

The space of connections $\mathcal{A}$ is an affine space modelled over $\Omega^1(M,\operatorname{ad}P)$. Hence, we have the identification $T_A\mathcal{A}\simeq\Omega^1(M,\operatorname{ad}P)$ for every $A\in\mathcal{A}$. Also, $C(P)$ is an affine bundle modelled over the vector bundle $T^*M\otimes\operatorname{ad}P$. So, for every $a\in\Omega^1(M,\operatorname{ad}P)\subset\Gamma(C(P),T^*M\otimes\operatorname{ad}P)$ we have a vertical vector field $X_a\in\mathfrak{X}^v(C(P))$.

Lemma 2. *For every $a,b\in\Omega^1(M,\operatorname{ad}P)$, we have*

$$i_{X_a}\mathbb{F}=p^*a,\qquad i_{X_b}i_{X_a}\mathbb{F}=0.$$

Proof. It follows from the formula (5.8) in [10]. □

If $A_0,A_1\in\mathcal{A}$, define $A_t=(1-t)A_0+tA_1$ and $a=A_1-A_0\in\Omega^1(M,\operatorname{ad}P)$. The tangent vector to the curve $\sigma_{A_t}(x)$ in $C(P)$ is $X_a(\sigma_{A_t}(x))$ for any $x\in M$, and hence we recover the usual transgression formula

$$c_f(F_{A_1})-c_f(F_{A_0})=d\left(\int_0^1\sigma_{A_t}^*(i_{X_a}c_f(\mathbb{F}))\,\mathrm{d}t\right)=d\left(k\int_0^1 f(a,F_{A_t},\dots,F_{A_t})\,\mathrm{d}t\right).$$

Given a connection A_0 on P, $\bar{p}^*A_0$ is a connection on $\mathbb{P}$. As $\bar{p}^*A_0$ and $\mathbb{A}$ are connections on the same bundle, defining $a_0=\mathbb{A}-\bar{p}^*A_0\in\Omega^1(C(P),\mathfrak{g})$, $A_t=(1-t)\bar{p}^*A_0+t\mathbb{A}$ and

$$\eta_f(A_0)=k\int_0^1 f(a_0,F_{A_t},\dots,F_{A_t})\,\mathrm{d}t,$$

we have $c_f(\mathbb{F})-c_f(F_{\bar{p}^*A_0})=\mathrm{d}\eta_f(A_0)$. If $2k>n$, then $c_f(F_{\bar{p}^*A_0})=p^*c_f(F_{A_0})=0$, and hence $c_f(\mathbb{F})=\mathrm{d}\eta_f(A_0)$.

3. Equivariant characteristic forms

First, we recall the definition of equivariant cohomology in the Cartan model (e.g. see [5,19]). Suppose that we have a left action of a connected Lie group $\mathcal{G}$ on a manifold N, i.e. a homomorphism $\rho:\mathcal{G}\to\operatorname{Diff}(N)$. We have an induced Lie algebra homomorphism

$$\operatorname{Lie}\mathcal{G}\to\mathfrak{X}(N),\qquad X\mapsto X_N=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\rho(\exp(-tX)).$$

Let $\Omega_{\mathcal{G}}(N)=(\boldsymbol{S}^\bullet(\operatorname{Lie}\mathcal{G}^*)\otimes\Omega^\bullet(N))^{\mathcal{G}}=\mathcal{P}^\bullet(\operatorname{Lie}\mathcal{G},\Omega^\bullet(N))^{\mathcal{G}}$ be the space of $\mathcal{G}$-invariant polynomials on $\operatorname{Lie}\mathcal{G}$ with values in $\Omega^\bullet(N)$. We define the following graduation: $\deg(\alpha)=$

$2k + r$ if $\alpha \in \mathcal{P}^k(\mathrm{Lie}\,\mathcal{G}, \Omega^r(N))$. Hence the space of $\mathcal{G}$-equivariant differential q-forms is

$$\Omega^q_{\mathcal{G}}(N) = \bigoplus_{2k+r=q} (\mathcal{P}^k(\mathrm{Lie}\,\mathcal{G}, \Omega^r(N)))^{\mathcal{G}}.$$

Let $d_c : \Omega^q_{\mathcal{G}}(N) \to \Omega^{q+1}_{\mathcal{G}}(N)$ be the Cartan differential

$$(d_c\alpha)(X) = d(\alpha(X)) - i_{X_N}\alpha(X), \quad \forall X \in \mathrm{Lie}\,\mathcal{G}.$$

As is well known, on $\Omega^\bullet_{\mathcal{G}}(N)$ we have $(d_c)^2 = 0$. Moreover, the equivariant cohomology (in the Cartan model) of N with respect of the action of $\mathcal{G}$ is defined as the cohomology of this complex, i.e.,

$$H^q_{\mathcal{G}}(N) = \frac{\ker(d_c : \Omega^q_{\mathcal{G}}(N) \to \Omega^{q+1}_{\mathcal{G}}(N))}{\mathrm{Im}(d_c : \Omega^{q-1}_{\mathcal{G}}(N) \to \Omega^q_{\mathcal{G}}(N))}.$$

Definition 3. Given a closed and $\mathcal{G}$-invariant form $\omega \in \Omega^q(M)$, an equivariant differential form $\omega^\# \in \Omega^q_{\mathcal{G}}(M)$ is said to be a $\mathcal{G}$-equivariant extension of ω if $d_c\omega^\# = 0$ and $\omega^\#(0) = \omega$.

In general, there could be obstructions to the existence of equivariant extensions (e.g., see [24]) but, as we will see, the classical construction of equivariant characteristic classes really provides canonical equivariant extensions for the forms we are dealing with.

Next, let us recall the relationship between equivariant cohomology and the cohomology of the quotient space. If the action of $\mathcal{G}$ on N is free and $N/\mathcal{G}$ is a manifold, then $N \to N/\mathcal{G}$ is a principal $\mathcal{G}$-bundle. Let A be a connection on this bundle. The following map is a generalization of the Chern-Weil homomorphism:

$$\mathrm{ChW}_A : \Omega^\bullet_{\mathcal{G}}(N) \to (\Omega^\bullet(N))_{\mathrm{basic}} \simeq \Omega^\bullet\left(\frac{N}{\mathcal{G}}\right), \qquad \alpha \mapsto (\alpha(F_A))_{\mathrm{hor}},$$

where β_{hor} denotes the horizontal component of $\beta \in \Omega^\bullet(N)$ with respect to the connection A. We have the following proposition.

Proposition 4. *If* $\alpha \in \Omega^\bullet_{\mathcal{G}}(N)$, *then* $\mathrm{ChW}_A(d_c\alpha) = d(\mathrm{ChW}_A(\alpha))$.

Proof. We refer the reader to [5, Theorem 7.34]. □

Theorem 5. *The induced map in cohomology* $\mathrm{ChW}_A : H^\bullet_{\mathcal{G}}(N) \to H^\bullet(N/\mathcal{G})$ *is independent of the connection A chosen, and is denoted by*

$$\mathrm{ChW}_N : H^\bullet_{\mathcal{G}}(N) \to H^\bullet\left(\frac{N}{\mathcal{G}}\right).$$

Proof. The result quickly follows by working on the bundle of connections. We use the notations introduced in Section 2 by setting $P = N$, $M = N/\mathcal{G}$ and denoting by $p : P \to M$ the quotient map. Let $\alpha \in \Omega^q_{\mathcal{G}}(P)$ be an equivariant q-form such that $d_c\alpha = 0$. Recall that $\overline{p}^*(\alpha)$ belongs to $\Omega^q_{\mathcal{G}}(\mathbb{P})$ as $\overline{p}$ is a $\mathcal{G}$-equivariant map. By Proposition 4, the form $\mathrm{ChW}_{\mathbb{A}}(\overline{p}^*\alpha) \in \Omega^\bullet(C(P))$ is closed, and from the formula (2) we obtain

$$\sigma^*_A(\mathrm{ChW}_{\mathbb{A}}(\overline{p}^*\alpha)) = \mathrm{ChW}_A(\alpha).$$

Again the result follows as the space of connections is contractible. □

Remark 6. If $\mathcal{G}$ is compact and connected ChW_N is an isomorphism (e.g. see [19]).

The definition of equivariant characteristic classes of Berline and Vergne (see [6,7,9]) can be introduced as follows. Let $\pi : P \to M$ a principal G-bundle and let us further assume that a Lie group $\mathcal{G}$ acts (on the left) on P by automorphisms of this bundle. Let A be a connection on P, which is *invariant under the action of* $\mathcal{G}$.

For every $f \in \mathcal{I}^G_k$ the $\mathcal{G}$-equivariant characteristic form associated to f and A, $c_f(F^{\mathcal{G}}_A) \in \Omega^{2k}_{\mathcal{G}}(M)$, is defined by

$$\begin{aligned} c_f(F^{\mathcal{G}}_A)(X) &= f\left(F_A - A(X_P), \overset{(k}{\ldots}, F_A - A(X_P)\right) \\ &= \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} f(F_A, \overset{(i}{\ldots}, F_A, A(X_P), \overset{(k-i}{\ldots}, A(X_P)) \end{aligned}$$

for every $X \in \mathrm{Lie}\,\mathcal{G}$.

Proposition 7. *We have*

(1) $c_f(F^{\mathcal{G}}_A)$ *is a* $\mathcal{G}$*-equivariant extension of* $c_f(F_A)$.
(2) *The equivariant cohomology class* $c^{\mathcal{G}}_f(P) = [c_f(F^{\mathcal{G}}_A)] \in H^{2k}_{\mathcal{G}}(M)$ *is independent of the* $\mathcal{G}$*-invariant connection* A*, and is called the* $\mathcal{G}$*-equivariant cohomology class of* P *associated to* f.

Proof. See [9,7]. □

Applying the construction of equivariant characteristic forms to the bundle $\mathbb{P} \to C(P)$ with the $\mathrm{Aut}\,P$-invariant connection $\mathbb{A}$, we obtain the $\mathrm{Aut}\,P$-equivariant characteristic form $c_f(\mathbb{F}^{\mathrm{Aut}\,P}) \in \Omega^{2k}_{\mathrm{Aut}\,P}(C(P))$, with an $\mathrm{Aut}\,P$-equivariant extension of $c_f(\mathbb{F})$. If $\mathcal{G} \subset \mathrm{Aut}\,P$ is any subgroup of the automorphism group, we have the corresponding $\mathcal{G}$-equivariant characteristic form

$$c_f(\mathbb{F}^{\mathcal{G}}) = c_f(\mathbb{F}^{\mathrm{Aut}\,P})|_{\mathrm{Lie}\,\mathcal{G}}.$$

The following proposition easily follows from the formula (2).

Proposition 8. *If $\mathcal{G}$ acts on $\pi : P \to M$ by automorphisms of π, and A is a $\mathcal{G}$-invariant connection on P, then we have $\sigma_A^*(c_f(\mathbb{F}^{\mathcal{G}})) = c_f(F_A^{\mathcal{G}})$.*

Remark 9. In this way, we obtain the analogous situation to that of the ordinary characteristic classes; see the last paragraph in Section 2. Moreover, Proposition 8 provides a very simple proof of Proposition 7(2), as the space of $\mathcal{G}$-invariant connections is an affine subspace; more precisely, if A is a $\mathcal{G}$-invariant connection, σ_A^* is the inverse of $p^* : H^\bullet_{\mathcal{G}}(M) \to H^\bullet_{\mathcal{G}}(C(P))$ (hence p^* is an isomorphism). Under this isomorphism the $\mathcal{G}$-equivariant cohomology class of $c_f(\mathbb{F}^{\mathcal{G}})$ corresponds to the $\mathcal{G}$-equivariant characteristic class associated to f. Moreover, as in the case of ordinary characteristic classes, the equivariant characteristic forms contain more information than their corresponding characteristic classes. For example, in Section 5 we will use this forms in the case $\mathcal{G} = \mathrm{Gau}\, P$ to find equivariant extensions of the forms (1).

The analog of Proposition 5 for the equivariant characteristic classes, is the following proposition.

Proposition 10. *Assume that $\mathcal{G}$ acts freely on P and M, and that the quotient bundle $\pi_{\mathcal{G}} : P/\mathcal{G} \to M/\mathcal{G}$ exists, then*

$$\mathrm{ChW}_M(c_f^{\mathcal{G}}(P)) = c_f\left(\frac{P}{\mathcal{G}}\right).$$

Proof. We denote by $q_P : P \to P/\mathcal{G}, q_M : M \to M/\mathcal{G}$ the projections. Let A_1 a connection on the principal G-bundle $\pi_{\mathcal{G}} : P/\mathcal{G} \to M/\mathcal{G}$, and A_2 a connection in the principal $\mathcal{G}$-bundle $M \to M/\mathcal{G}$. Clearly $A_1' = q_P^* A_1$ is a $\mathcal{G}$-invariant connection on the principal G-bundle $P \to M$, and for every $X \in \mathrm{Lie}\,\mathcal{G}$ we have $A_1'(X_P) = 0$. So, the equivariant characteristic class associated to A_1' and f is the basic form $c_f(F^{\mathcal{G}}_{A_1'}) = c_f(F_{A_1'})$.

From the very definition of ChW_{A_2}, it is clear that

$$\mathrm{ChW}_{A_2}(c_f(F_{A_1'})) = c_f(F_{A_1}),$$

and hence the result follows. □

4. Forms in $\Gamma(E)$ induced by forms in $J^r E$

Let $q : E \to M$ be a locally trivial bundle over an oriented, connected, and compact n-manifold without boundary M. We denote by $\Gamma(E)$ the space of global sections of E, and we assume that it is not empty. We consider $\Gamma(E)$ as a differential manifold; for the details of its infinite-dimensional structure, see [20]. For any $s \in \Gamma(E)$ there is an identification $T_s\Gamma(E) \simeq \Gamma(M, s^*V(E))$. We denote by $J^r E$ the r-jet bundle of E, and by $\mathrm{Proj}(E)$ the group of projectable diffeomorphisms of E, i.e. $\phi \in \mathrm{Diff}(E)$ such that there exist $\underline{\phi} \in \mathrm{Diff}(M)$ with $q \circ \phi = \underline{\phi} \circ q$. We denote by $\mathrm{Proj}^+(E)$ the subgroup of elements $\phi \in \mathrm{Proj}(E)$ such that $\underline{\phi} \in \mathrm{Diff}^+(M)$, the group of orientation preserving diffeomorphisms.

We denote by $\mathrm{proj}(E) \subset \mathfrak{X}(E)$ the Lie algebra of projectable vector fields, which can be considered as the Lie algebra of $\mathrm{Proj}(E)$. The group $\mathrm{Proj}(E)$ acts on $\Gamma(E)$ by $(\phi, s) \mapsto \phi_{\Gamma(E)}(s) = \phi \circ s \circ \underline{\phi}^{-1}$. At the Lie-algebra level, every projectable vector field $X \in \mathrm{proj}(E)$ determines a vector field $X_{\Gamma(E)} \in \mathfrak{X}(\Gamma(E))$ on $\Gamma(E)$. Given $\phi \in \mathrm{Proj}(E)$ (resp. $X \in \mathrm{proj}(E)$) we denote by $\phi^{(r)}$ (resp. $X^{(r)}$) its prolongation to $J^r(E)$. We recall that $\phi^{(r)}(j^r_x s) = j^r_{\underline{\phi}(x)}(\phi_{\Gamma(E)}(s))$.

The evaluation map

$$\mathrm{ev}_r : M \times \Gamma(E) \to J^r E, \qquad (x, s) \mapsto j^r_x s$$

is equivariant with respect of the action of $\mathrm{Proj}(E)$ on $M \times \Gamma(E)$ and $J^r E$. So, for any $X \in \mathrm{proj}(E)$, denoting by $\underline{X} \in \mathfrak{X}(E)$ its projection to M, we have

$$\mathrm{ev}_{r*}(\underline{X}, X_{\Gamma(E)}) = X^{(r)}. \tag{3}$$

We define a map

$$\mathcal{F} : \Omega^{n+k}(J^r E) \to \Omega^k(\Gamma(E))$$

by the formula

$$\mathcal{F}[\alpha] = \int_M \mathrm{ev}_r^* \alpha \in \Omega^k(\Gamma(E)), \tag{4}$$

where $\int_M$ denotes the integration over the fiber of $M \times \Gamma(E) \to \Gamma(E)$. If $\alpha \in \Omega^k(J^r E)$ with $k < n$,we set $\mathcal{F}[\alpha] = 0$.

Proposition 11. *For any $\alpha \in \Omega^{n+k}(J^r E)$, we have*

$$(\mathcal{F}[\alpha])_s(X_1, \ldots, X_k) = \int_M (j^r s)^*(i_{X_k^{(r)}} \ldots i_{X_1^{(r)}} \alpha) \tag{5}$$

for every $s \in \Gamma(E)$, $X_1, \ldots, X_k \in T_s\Gamma E \simeq \Gamma(M, s^*V(E))$.

Proof. The result follows from the definition of $\mathcal{F}[\alpha]$ and the formula (3) applied to vertical vector fields. □

The following proposition follows from the definition of $\mathcal{F}$ and the properties of the integration over the fiber.

Proposition 12. *For every $\alpha \in \Omega^{n+k}(J^r E)$, $\phi \in \mathrm{Proj}^+(E)$, and $X \in \mathrm{proj}(E)$ we have*

(a) $\mathcal{F}[d\alpha] = \mathrm{d}\mathcal{F}[\alpha]$,
(b) $\mathcal{F}[(\phi^{(r)})^* \alpha] = \phi^*_{\Gamma(E)} \mathcal{F}[\alpha]$,
(c) $\mathcal{F}[i_{X^{(r)}} \alpha] = \mathrm{i}_{X_{\Gamma(E)}} \mathcal{F}[\alpha]$,
(d) $\mathcal{F}[L_{X^{(r)}} \alpha] = \mathrm{L}_{X_{\Gamma(E)}} \mathcal{F}[\alpha]$.

Remark 13. If $\alpha \in \Omega^{n-1}(J^r E)$, the condition 12(a) means $\mathcal{F}[d\alpha] = 0$.

Now, assume that $\mathcal{G}$ is a subgroup of $\mathrm{Proj}^+(E)$. If $\alpha \in \mathcal{P}^q(\mathrm{Lie}\,\mathcal{G}, \Omega^{n+k}(E))$, the composition

$$\mathrm{Lie}\,\mathcal{G} \xrightarrow{\alpha} \Omega^{n+k}(J^r E) \xrightarrow{\mathcal{F}} \Omega^k(\Gamma(E))$$

defines an element $\mathcal{F}[\alpha]$ of $\mathcal{P}^q(\mathrm{Lie}\,\mathcal{G}, \Omega^k(\Gamma(E)))$; that is, for $X \in \mathrm{Lie}\,\mathcal{G}$ we have

$$(\mathcal{F}[\alpha])(X) = \mathcal{F}[\alpha(X)].$$

By Proposition 12(b) if α is $\mathcal{G}$-invariant, $\mathcal{F}[\alpha]$ is also $\mathcal{G}$-invariant, and so the map $\mathcal{F}$ extends to a map between $\mathcal{G}$-equivariant differential forms

$$\mathcal{F} : \Omega^{n+k}_{\mathcal{G}}(J^r E) \to \Omega^k_{\mathcal{G}}(\Gamma(E)).$$

Proposition 14. *For every* $\alpha \in \Omega^{n+k}_{\mathcal{G}}(J^r E)$ *we have* $\mathcal{F}[d_c\alpha] = \mathrm{d}_c\mathcal{F}[\alpha]$. *Hence, we have an induced map in equivariant cohomology* $\mathcal{F} : H^{n+k}_{\mathcal{G}}(J^r E) \to H^k_{\mathcal{G}}(\Gamma(E))$.

Proof. If $\alpha \in \Omega^{n+k}_{\mathcal{G}}(J^r E)$ and $X \in \mathrm{Lie}\,\mathcal{G}$, then from Proposition 12 we have

$$\begin{aligned}(\mathcal{F}[d_c\alpha])(X) &= \mathcal{F}[d_c\alpha(X)] = \mathcal{F}[d(\alpha(X))] - \mathcal{F}[i_{X^{(r)}}\alpha(X)] \\ &= \mathrm{d}\mathcal{F}[\alpha(X)] - \mathrm{i}_{X_{\Gamma(E)}}\mathcal{F}[\alpha(X)] = (\mathrm{d}_c\mathcal{F}[\alpha])(X). \quad \square\end{aligned}$$

5. Applications

In this section we combine the results of Sections 3 and 4. As remarked in Section 1, in Gauge theories Gau P-invariant forms are specially interesting, so we focus on these forms. In [11] it is proved that the space of Gau P-invariant forms is generated by the forms of type $c_f(\mathbb{F}) \wedge p^*\beta$, with $f \in \mathcal{I}^G_k$ and $\beta \in \Omega^r(M)$. We assume that β is closed and $2k + r \geq n$.

By applying the map $\mathcal{F}$ to $c_f(\mathbb{F}) \wedge p^*\beta$ we obtain

$$C_{f,\beta} = \mathcal{F}[c_f(\mathbb{F}) \wedge p^*\beta] \in \Omega^{2k+r-n}(\mathcal{A}).$$

By Proposition 12 this form is closed and Gau P-invariant.

Taking Lemma 2 into account, it is easy to obtain the expression of $C_{f,\beta}$. We have the following proposition.

Proposition 15. *Let* $q = 2k + r - n$. *For* $a_1, \ldots, a_q \in \Omega^1(M, \mathrm{ad}\,P)$ *we have*:

$$(C_{f,\beta})_A(a_1, \ldots, a_q) = \binom{k}{q} \int_M f(a_1, \ldots, a_q, F_A, \overset{(n-k-r}{\ldots}, F_A) \wedge \beta.$$

By virtue of Proposition 7, the $\mathrm{Gau}\,P$-equivariant characteristic form $c_f(\mathbb{F}^{\mathrm{Gau}\,P})$ is an equivariant extension of $c_f(\mathbb{F})$. Also $p^*\beta$ is a closed $\mathrm{Gau}\,P$-equivariant differential form, because it is closed and basic. So $c_f(\mathbb{F}^{\mathrm{Gau}\,P}) \wedge p^*\beta$ is a $\mathrm{Gau}\,P$-equivariant extension of $c_f(\mathbb{F}) \wedge p^*\beta$. We thus obtain the following theorem.

Theorem 16. *The* $\mathrm{Gau}\,P$*-equivariant form*

$$C^{\#}_{f,\beta} = \mathcal{F}[c_f(\mathbb{F}^{\mathrm{Gau}\,P}) \wedge p^*\beta] \in \Omega^{2k+r-n}_{\mathrm{Gau}\,P}(\mathcal{A}) \tag{6}$$

is a $\mathrm{Gau}\,P$*-equivariant extension of* $C_{f,\beta}$.

We have thus found a canonical $\mathrm{Gau}\,P$-equivariant extension of $C_{f,\beta}$, as said in Section 1.

Let $X \in \mathrm{gau}\,P = \Omega^0(M, \mathrm{ad}\,P) \simeq \Omega^0_{\mathrm{Ad}}(P, \mathfrak{g})$, and $X_{\mathbb{P}} \in \mathfrak{X}(\mathbb{P})$ the vector field corresponding to the action of $\mathrm{Gau}\,P$ in $\mathbb{P}$. We have

$$\mathbb{A}_{(\sigma_A(x),u)}(X_{\mathbb{P}}) = A_u(\bar{p}_* X_{\mathbb{P}}) = A_u(X_P) = X(u).$$

So, $\mathbb{A}(X_{\mathbb{P}}) = p^*X$, and hence

$$c_f(\mathbb{F}^{\mathrm{Gau}\,P})(X) = f\left(\mathbb{F} - p^*X, \overset{(k}{\ldots}, \mathbb{F} - p^*X\right) = \sum_{i=0}^{k} c^i_f(\mathbb{F}, X), \tag{7}$$

where $c^i_f(\mathbb{F}, X) = (-1)^i \binom{k}{i} f(\mathbb{F}, \ldots^{(k-i}, \mathbb{F}, p^*X, \ldots^{(i}, p^*X)$.

The condition $d_c c_f(\mathbb{F}^{\mathrm{Gau}\,P}) = 0$ is equivalent to

$$dc_f(\mathbb{F}) = 0, \qquad dc^i_f(\mathbb{F}, X) = i_{X_{C(P)}} c^{i-1}_f(\mathbb{F}, X), \quad i = 1, \ldots, k. \tag{8}$$

Using (7) and Proposition 15 it is easy to obtain the expression for $C^{\#}_{f,\beta}$. In Example 19 of Section 5.2, we detail such expression in a simple case. Also, for $f(X) = \mathrm{Tr}(\exp X)$ we obtain the equivariant forms defined in [21, Section 6] as a particular case.

5.1. Forms in $\mathcal{A}/\mathrm{Gau}^0 P$

Let $\mathrm{Gau}^0 P \subset \mathrm{Gau}\,P$ be the subgroup of gauge transformations acting as the identity on the fiber over a fixed point $x_0 \in M$. Then $\mathrm{Gau}^0 P$ acts freely on $\mathcal{A}$ and the quotient $\mathcal{A}/\mathrm{Gau}^0 P$ is well defined (e.g. see [23,12]).

By virtue of Theorem 5, the $\mathrm{Gau}^0 P$-equivariant differential form $C^{\#}_{f,\beta}$ determines a cohomology class

$$\mathrm{ChW}_{\mathcal{A}}(C^{\#}_{f,\beta}) \in H^{2k+r-n}\left(\frac{\mathcal{A}}{\mathrm{Gau}^0 P}\right).$$

To obtain a representative of this class it is necessary a connection on the bundle $\mathcal{A} \to \mathcal{A}/\mathrm{Gau}^0 P$. The construction of this connection is a standard fact in gauge theories (e.g. see

[12]): given a Riemannian metric g in M there is a connection in $\mathcal{A} \to \mathcal{A}/\mathrm{Gau}^0 P$ given by the decomposition

$$T_A\mathcal{A} \simeq \Omega^1(M, \mathrm{ad}\, P) \simeq \mathrm{Im}(d_A) \oplus \ker(d_A^*).$$

The space $\mathrm{Im}(d_A)$ is the tangent space to the orbits, and $\ker(d_A^*)$ is the horizontal complement. The expression of its corresponding connection form is

$$\mathfrak{A} = G_A d_A^*, \tag{9}$$

where $G_A = (d_A^* \circ d_A)^{-1}$ is the Green function of the Laplacian

$$\Delta_A^0 = d_A^* \circ d_A : \Omega^0(M, \mathrm{ad}\, P) \to \Omega^0(M, \mathrm{ad}\, P).$$

We denote by $\mathfrak{F}$ the curvature of $\mathfrak{A}$.

Next, we relate our classes $\mathrm{ChW}_{\mathcal{A}}(C^{\#}_{f,\beta})$ to the constructions in [3]. Consider the principal G-bundle $P \times \mathcal{A} \to M \times \mathcal{A}$. The group $\mathrm{Gau}^0 P$ acts on P and on $\mathcal{A}$. Taking the quotient, we obtain a principal G-bundle

$$\mathcal{Q} = \frac{P \times \mathcal{A}}{\mathrm{Gau}^0 P} \to M \times \left(\frac{\mathcal{A}}{\mathrm{Gau}^0 P}\right).$$

If $f \in \mathcal{I}_k^G$ and $[\beta] \in H^r(M)$, we have a class

$$c_f(\mathcal{Q}) \wedge [\beta] \in H^{2k+r}\left(M \times \left(\frac{\mathcal{A}}{\mathrm{Gau}^0 P}\right)\right).$$

Integrating over M we obtain the class

$$\mu_f([\beta]) = \int_M c_f(\mathcal{Q}) \wedge [\beta] \in H^{2k+r-n}\left(\frac{\mathcal{A}}{\mathrm{Gau}^0 P}\right).$$

Theorem 17. *For every $f \in \mathcal{I}_k^G$ and every closed $\beta \in \Omega^r(M)$ with $2k + r \geq n$, we have*

$$\mathrm{ChW}_{\mathcal{A}}(C^{\#}_{f,\beta}) = \mu_f([\beta]).$$

Proof. The evaluation map extends to a morphism of principal G-bundles

$$\begin{array}{ccc} P \times \mathcal{A} & \xrightarrow{\overline{\mathrm{ev}}} & \mathbb{P} \\ \downarrow & & \downarrow \bar{\pi} \\ M \times \mathcal{A} & \xrightarrow{\mathrm{ev}} & C(P) \end{array}$$

where $\overline{\mathrm{ev}}(u, A) = (\sigma_A(x), u)$ for $u \in \pi^{-1}(x)$. Hence, $\overline{\mathrm{ev}}^*(\mathbb{A})$ is a connection on $P \times \mathcal{A} \to M \times \mathcal{A}$, $\mathrm{ev}^*(c_f(\mathbb{F}))$ gives its characteristic classes, and $\mathrm{ev}^*(c_f(\mathbb{F}^{\mathrm{Gau}^0 P}))$, $f \in \mathcal{I}_k^G$, are its

$\mathrm{Gau}^0 P$-equivariant characteristic forms. By Proposition 10 these equivariant characteristic forms determine the characteristic classes of the quotient bundle $\mathcal{Q} \to M \times (\mathcal{A}/\mathrm{Gau}^0 P)$, i.e., we have

$$\mathrm{ChW}_{M\times\mathcal{A}}(\mathrm{ev}^*(c_f(\mathbb{F}^{\mathrm{Gau}^0 P}))) = c_f(\mathcal{Q}).$$

As the connection on $M \times \mathcal{A} \to M \times (\mathcal{A}/\mathrm{Gau}^0 P)$ is given by the connection (9), the cohomology class $\mu_f([\beta])$, is represented by the form

$$\left(\int_M c_f(\mathrm{ev}^*(\mathbb{F}) - \mathfrak{F}) \wedge \beta\right)_{\mathrm{hor}}. \tag{10}$$

From (4) and (7) we have

$$C^\#_{f,\beta}(X) = \int_M \mathrm{ev}^*(c_f(\mathbb{F}^{\mathrm{Gau}^0 P}(X)) \wedge p^*\beta) = \int_M c_f(\mathrm{ev}^*(\mathbb{F}) - X) \wedge \beta.$$

So $\mathrm{ChW}_{\mathcal{A}}(C^\#_{f,\beta})$ is also represented by the form (10). □

Remark 18. The construction of the classes $\mu_f([\beta])$ appears in [3] in order to compute the Chern character of the index of families of Dirac operators and to apply them to the study of anomalies in gauge theories. These classes also appear in other constructions in gauge theories, like the definition of Donaldson invariants (see [13]), Topological Quantum Field Theory ([8,4]), etc.

It is remarkable that we obtain these classes only by studying $\mathrm{Gau}\,P$-invariant forms on $C(P)$ and its equivariant extensions.

5.2. Moment maps

Example 19. Let M be a surface, and $G = U(k)$. If $f(X) = (1/8\pi^2)\mathrm{tr}(X^2)$, $X \in \mathfrak{g}$, then the corresponding characteristic class on M vanishes by dimensional reasons, but the characteristic form $c_f(\mathbb{F}) \in \Omega^4(C(P))$ does not. From our constructions, this form defines a closed and $\mathrm{Gau}\,P$-invariant two-form on $\mathcal{A}$, $C_f = \mathcal{F}[c_f(\mathbb{F})]$. By Proposition 15 and formula (7) for $a, b \in \Omega^1(M, \mathrm{ad}\,P)$, and $X \in \mathrm{gau}\,P = \Omega^0(M, \mathrm{ad}\,P)$, we have

$$c_f(\mathbb{F}) = \frac{1}{8\pi^2}\mathrm{tr}(\mathbb{F} \wedge \mathbb{F}),$$

$$c_f(\mathbb{F}^{\mathrm{Gau}\,P})(X) = \frac{1}{8\pi^2}\mathrm{tr}(\mathbb{F} \wedge \mathbb{F}) - \frac{1}{4\pi^2}\mathrm{tr}(p^*X \cdot \mathbb{F}) + \frac{1}{8\pi^2}\mathrm{tr}((p^*X)^2),$$

$$(C_f)_A(a, b) = \frac{1}{4\pi^2}\int_M \mathrm{tr}(a \wedge b),$$

$$(C^\#_f(X))_A(a, b) = \frac{1}{4\pi^2}\int_M \mathrm{tr}(a \wedge b) - \frac{1}{4\pi^2}\int_M \mathrm{tr}(X \cdot F_A).$$

Hence, C_f coincides with the natural symplectic structure on $\mathcal{A}$ defined in [1]. Moreover, in the case of a two-form, it is equivalent to give an equivariant extension of the form and a moment map (e.g. see [2]). Hence $C_f^{\#}$ defines a canonical moment map m for this symplectic structure, given by

$$m : \mathcal{A} \to (\text{gau}\, P)^*, \qquad m_A(X) = -\frac{1}{4\pi^2} \int_M \text{tr}(X \cdot F_A).$$

Under the pairing

$$\Omega^2(M, \text{ad}\, P) \times \Omega^0(M, \text{ad}\, P) \to \mathbb{R}, \qquad (\eta, X) \mapsto \langle \eta, X \rangle = -\frac{1}{4\pi^2} \int_M \text{tr}(X \cdot \eta) \tag{11}$$

this moment map corresponds to the curvature F_A, and it thus coincides with that defined in [1].

Also, we have $m^{-1}(0) = \{A \in \mathcal{A} : F_A = 0\}$, and by symplectic reduction we obtain the moduli space of flat connections, and our form gives rise to the symplectic structure on this space.

More generally, let (M, σ) be a symplectic $2n$-manifold. Then the form

$$\frac{1}{(n-1)!} c_f(\mathbb{F}) \wedge \sigma^{n-1} \in \Omega^{2n+2}(C(P))$$

defines a symplectic structure on $\mathcal{A}$, and the equivariant extension provides a moment map for it, which, in particular, coincides with that obtained in [13, Proposition 6.5.8] and [21, Section 3].

5.3. Chern–Simons terms

Suppose that M has dimension $2k-1$ and $f \in \mathcal{I}_k^G$. Then $c_f(\mathbb{F}) \in \Omega^{2k}(C(P))$ defines a first order locally variational operator (see [14]). Let $h : \Omega^\bullet \to \Omega^\bullet(J^1C(P))$ denote the horizontalization operator. As we have $c_f(\mathbb{F}) = d\eta_f(A_0)$, the form $h(\eta_f(A_0)) \in \Omega^{2k-1}(J^1(C(P)))$, is a Lagrangian for this operator, and hence this operator is globally variational.

We know that $c_f(\mathbb{F})$ is Gau P-invariant, but $\eta_f(A_0)$ is not invariant, because it depends on the connection A_0. However, by virtue of (8) for every $X \in \text{gau}\, P$, we have

$$\begin{aligned} L_{X_{C(P)}}\eta_f(A_0) &= i_{X_{C(P)}} d\eta_f(A_0) + d i_{X_{C(P)}}\eta_f(A_0) = i_{X_{C(P)}} c_f(\mathbb{F}) + d i_{X_{C(P)}}\eta_f(A_0) \\ &= d(c_f^1(\mathbb{F}, X) + i_{X_{C(P)}}\eta_f(A_0)), \end{aligned}$$

and hence $L_{X_{C(P)}}\eta_f(A_0)$ is exact. As it is shown in [18] this condition leads to a Noether conservation law. In fact, the conserved current is $\mathfrak{J}(X) = h(c_f^1(\mathbb{F}, X))$, because by the results in [14] A is an extremal connection if and only if $\sigma_A^*(i_Y c_f(\mathbb{F})) = 0, \forall Y \in \mathfrak{X}(C(P))$, and in this case, for any $X \in \mathrm{gau}\,P$, we have

$$d\sigma_A^*(c_f^1(\mathbb{F}, X)) = \sigma_A^* d(c_f^1(\mathbb{F}, X)) = \sigma_A^*(i_{X_{C(P)}} c_f(\mathbb{F})) = 0.$$

More generally, if $f \in \mathcal{I}_k^G$, the form $\beta \in \Omega^r(M)$ is closed, and $\dim(M) = 2k + r - 1$, the form $c_f(\mathbb{F}) \wedge p^*\beta$ defines a first order globally variational operator with Lagrangian density $\lambda = h(\eta_f(A_0)) \wedge p^*\beta$ and with conserved current $\mathfrak{J}(X) = h(c_f^1(\mathbb{F}, X)) \wedge p^*\beta$.

The form $c_f(\mathbb{F})$ defines a closed and $\mathrm{Gau}\,P$-invariant one-form $\mathcal{F}[c_f(\mathbb{F})]$ on the space of connections $\mathcal{A}$. This form is also horizontal, because for every $X \in \mathrm{gau}\,P$ we have

$$i_{X_\mathcal{A}} \mathcal{F}[c_f(\mathbb{F})] = \mathcal{F}[i_{X_{C(P)}} c_f(\mathbb{F})] = \mathcal{F}[dc_f^1(X, \mathbb{F})] = 0.$$

So, $\mathcal{F}[c_f(\mathbb{F})]$ projects to a closed one-form α_f on the space $\mathcal{A}/\mathrm{Gau}^0 P$. We have

$$\mathcal{F}[c_f(\mathbb{F})] = \mathcal{F}[d\eta_f(A_0)] = d\mathcal{F}[\eta_f(A_0)].$$

Hence $\mathcal{F}[c_f(\mathbb{F})]$ is the exterior differential of the function $\mathcal{F}[\eta_f(A_0)] \in \Omega^0(\mathcal{A})$. It is easy to see that the one-form $\alpha_f \in \Omega^1(\mathcal{A}/\mathrm{Gau}^0 P)$ is exact if and only if the function $\mathcal{F}[\eta_f(A_0)]$ is $\mathrm{Gau}^0 P$-invariant. We have

$$L_{X_\mathcal{A}} \mathcal{F}[\eta_f(A_0)] = \mathcal{F}[L_{X_C} \eta_f(A_0)] = \mathcal{F}[d(c_f^1(\mathbb{F}, X) + i_{X_C} \eta_f(A_0))] = 0,$$

and so this function is invariant under the action of the connected component with the identity in $\mathrm{Gau}\,P$. But in general it is not invariant under the action of the full group $\mathrm{Gau}^0 P$ (as it is shown in the following example), and in this case α_f defines a non-trivial cohomology class on $\mathcal{A}/\mathrm{Gau}^0 P$.

Example 20. Suppose that $G = \mathrm{SU}(2)$, $f = (1/4\pi^2)\det$ is the polynomial corresponding to the second Chern class and M is a three-manifold. Then the bundle P is trivial $P = M \times \mathrm{SU}(2)$, $\mathcal{A} \cong \Omega^1(M, \mathfrak{g})$ and $\mathrm{Gau}\,P = C^\infty(M, \mathrm{SU}(2))$. If A_0 is the connection corresponding to the product decomposition, then $\eta_f(A_0)$ is the classical Chern–Simons Lagrangian and for any $A \in \mathcal{A}$ we have

$$\mathcal{F}[\eta_f(A_0)]_A = \int_M \sigma_A^*(\eta_f(A_0)) = -\frac{1}{8\pi^2} \int_M \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

If $\varphi : M \to \mathrm{SU}(2)$ is a gauge transformation, it is a classical result (see [8]) that

$$\mathcal{F}[\eta_f(A_0)]_{\varphi \cdot A} = \mathcal{F}[\eta_f(A_0)]_A - S(\varphi),$$

where $S(\varphi)$ is the winding number of the map φ. As SU(2) is connected, every gauge transformation is homotopic to an element of $\mathrm{Gau}^0 P$. Hence there are elements $\varphi \in \mathrm{Gau}^0 P$ with $S(\varphi) \neq 0$, and $\mathcal{F}[\eta_f(A_0)]$ is not $\mathrm{Gau}^0 P$-invariant.

6. Concluding remarks

1. The Berline–Vergne definition of equivariant characteristic classes supposes that a $\mathcal{G}$-invariant connection is given. Note however that, independently of the existence of $\mathcal{G}$-invariant connections, the $\mathcal{G}$-equivariant characteristic forms always exist on $C(P)$ (since the canonical connection is $\mathcal{G}$-invariant), and the existence of $\mathcal{G}$-invariant connections is needed only in order to obtain $\mathcal{G}$-equivariant classes on M. We hope that our construction could be useful in the study of equivariant characteristic classes for non-compact Lie groups, where the existence of invariant connections is not guaranteed in general, and the analysis is much more involved, e.g., see [17].

   Moreover, in Section 5 we have used Gau P-equivariant characteristic forms. From the classical point of view of equivariant characteristic classes this procedure is meaningless as this group acts trivially on M and also there are no Gau P-invariant connections.
2. The usefulness of the map $\mathcal{F}$ lies in the fact that it provides a general procedure to obtain results about (equivariant) differential forms and cohomology classes on the infinite dimensional manifold $\Gamma(E)$ by working on a finite dimensional jet bundle. Note that, as in this paper we only consider forms on the 0-jet bundle (that is, on E), it could be thought that the consideration of jet bundles is unnecessary; but, for example in [15], we study the analogous results in the case of Riemannian metrics, and in this case we need to work with forms in the first jet bundle. In fact, there is a close relation between the map $\mathcal{F}$ and the variational bicomplex, that we will analyze in a forthcoming paper.

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